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Entropy Current Formalism for Supersymmetric Theories

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Abstract

The recent developments in fluid/gravity correspondence give a new impulse to the study of fluid dynamics of supersymmetric theories. In that respect, the entropy current formalism requires some modifications in order to be adapted to supersymmetric theories and supergravities. We formulate a new entropy current in superspace with the properties 1) its Hodge dual is a d-form in d space-dimensions, 2) it is conserved off-shell for non dissipative fluids, 3) it is invariant under rigid supersymmetry transformations 4) it is covariantly closed in local supersymmetric theories 5) it reduces to its bosonic expression on space-time. We compute the entropy density and, in the meanwhile, we define the Hodge duality for supermanifolds.

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1 Introduction

Recent developments in fluid/gravity correspondence [1, 2, 3] motivate a deeper analysis of the fluid dynamics in the context of supersymmetric theories and of supergravity. In the present work, we take a first step toward that extension by analyzing the definition of the entropy current for non dissipative fluids (see for example [4, 5, 6, 7, 8]) and by providing its supersymmetric generalization.[1] The starting point is a convenient formulation of the fluid dynamics in terms of the comoving coordinates of the fluid (see [10, 11]). The Eulerian description in terms of spacetime-dependent quantities is replaced by this new set of comoving coordinates ϕ^I (with $I = 1, \dots, d$, d being the space dimensions) which are spacetime fields. In terms of those, in the case of non-dissipative fluids, one can easily write down a Lagrangian whose field equations are the relativistic generalization of the well-known Navier-Stokes equations. One can also easily define several interesting thermodynamical quantities such as the entropy, the energy density, chemical potentials and so on. This formalism permits also a direct verification of Maxwell equations for thermodynamics. Finally, all techniques of quantum field theory can be used to investigate the quantum properties of fluids (see for example [5]).

Recently a series of interesting papers [12, 6, 13] appeared on the subject by exploring again the fluid dynamics from the point of view of comoving coordinates and discussing the role of the entropy current in that contest. In particular they claim that the entropy current of a given system must have the following properties: 1) It is dual to a d-form in $(d+1)$-space-time dimensions; 2) It is conserved off-shell. It is easy to show that the expression

$$J^{(1)} = \star_{(d+1)} d\phi^1 \wedge \cdots \wedge d\phi^d \,,$$

where $\star_{(d+1)}$ is the Hodge dual in $d+1$ dimensions, has the correct properties. In addition, it cannot be written as a d-exact expression since the comoving coordinates ϕ^I are not globally defined. The entropy density can be computed by considering the Hodge dual of J. In papers [6, 8, 7], this formalism has been applied to normal fluids as well as to superfluids and, there, all quantities are computed in terms of the comoving coordinates and of one additional degree of freedom ψ. A new symmetry has been advocated in order to describe the supefluid in a suitable phase and the spectrum of waves in that fluid have been taken into account. We briefly review that formalism in Section 2 in order to set up the stage for our developments.

The next step is to provide a supersymmetric extension. Since the coordinates ϕ^I represent a set of comoving coordinates of the fluid, it is natural to introduce a set of anticommuting coodinates θ^α for describing the fluid fermionic degrees of freedom (see [11]). Using the analogy with the Green-Schwarz superstring and with the supermembrane we define a supersymmetric 1-form Π^I replacing the 1-form $d\phi^I$ of the bosonic ones. In that context, we discuss the generalization of the action for supersymmetric fluids with all symmetries.

Finally, we can provide the supersymmetric extension of the entropy current. We have to recall that the entropy current is associated with volume preserving diffeomorphisms and therefore the supersymmetric extension must play a similar role of volume preserving superdiffeomorphisms. In that case the form is an *integral form* (see for example [14, 15, 16, 17, 18]) whose complete expression on the supermanifold $\mathcal{M}^{(d+1|m)}$ is

$$J^{(d+1|m)} = \frac{1}{d!} \epsilon_{I_1 \dots I_d} \Pi^{I_1} \wedge \cdots \wedge \Pi^{I_d} \wedge \delta(d\theta^1) \wedge \cdots \wedge \delta(d\theta^m) \,,$$

[1] For earlier works on supersymmetric description of fluids see for instance [9]

d being the spatial dimensions and m the dimension of the spinor representation, that is the number of fermionic coordinates. [2] This expression transforms as a Berezinian under superdiffeomorphisms and the Dirac delta of 1-superforms $d\theta^\alpha$ is a symbol that has the usual properties of distributions as explained in [17]. The Dirac delta-functions of 1-forms $d\theta^\alpha$ can be understood by assuming that the fermionic 1-forms are indeed commuting quantities and therefore it becomes pivotal to define an integration measure in this space. One way – although it is not the only one – is to use the atomic measure given by the distributional Dirac delta. The main property of that distribution is locality which plays an important role in our construction.

Given the new formula for the entropy current J, we can compute the entropy density s and we discuss some implications. This can be done by considering the Hodge-dual of the current $^\star J$ and by constructing the density $J \wedge {}^\star J$ using Hodge-duality on supermanifolds developed in the Appendix A3, which reduces, in the purely bosonic setting, to $s^2\, Vol_{d+1}$.

As an important application, we generalize our construction to supergravity. The paper is organized as follows:
In Section 2 we review the Lagrangian approach for fluid dynamics and introduce the entropy current.
In Section 3 we discuss the symmetries and equations of motion of a supersymmetric effective theory for fluid dynamics in a 1+1 dimensional model, introducing a supersymmetric entropy current for this model.
In Section 4 we extend the considerations of the previous section to a general $d+1$ dimensional Lagrangian and give a general expression for the entropy current and entropy density in superspace.
Finally, in Section 5 we give an expression for the entropy current in supergravity theories.
The Appendix contains several technical details.

2 Comoving Coordinate Formalism

In this section we shortly review the Lagrangian approach developed in ref.s [10, 4, 6, 8], which is based on the use of the comoving coordinates of the fluid as fundamental fields, adopting the same notations as [6]. Their approach will be useful for the extension of the formalism to the supersymmetric case.

From a physical point of view one assumes that the hydrodynamics of a perfect fluid can be formulated as a low energy effective Lagrangian of massless fields which are thought of as the Goldstone bosons of a broken symmetry, namely space translations (broken by the presence of phonons), and is invariant under the symmetry associated with conserved charges. The effective complete Lagrangian would be a derivative expansion in terms of the breaking parameters (mean free path and mean free time). One tries to determine the low energy Lagrangian by symmetry requirements.

Working, for the sake of generality, in $d+1$ space-time dimensions, one introduces d scalar fields $\phi^I(x^I,t)$, $I = 1,\ldots,d$ as Lagrangian comoving coordinates of a fluid element at the point x^I at time t such that, at equilibrium, the ground state is described by

[2] For the sake of clarity, we assume here and in the following that the θ^α are Majorana spinors (as it is in four dimensions) or Majorana-Weyl spinors. The case of Dirac or pseudo-Majorana spinors (as it happens e.g. in D=5 supersymmetric theories) can be dealt with in an analogous way.

$\Phi^I = x^I$ and requires, in absence of gravitation, the following symmetries:

$$\delta\phi^I = a^I \quad (a^I = \text{const.})\,, \tag{2.1}$$
$$\phi^I \to O^I_J\,\phi^J, \qquad (O^I_J \,\epsilon\, \mathrm{SO(d)})\,, \tag{2.2}$$
$$\phi^I \to \xi^I(\phi), \qquad \det(\partial\xi^I/\partial\phi^J) = 1. \tag{2.3}$$

Furthermore, if there is a conserved charge (particle number, electric charge etc.), then the associated symmetry cannot be described by transformations acting on the fields ϕ^I, since they are non compact and they cannot describe particle number conservation. Therefore one introduces a new field $\psi(x^I, t)$ which is a phase, that is it transforms under U(1) as follows

$$\psi \to \psi + c, \quad (c = \text{const.}). \tag{2.4}$$

Finally one must take into account that the particle number is comoving with the fluid, giving rise to a (matter) conserved current

$$\partial_\mu j^\mu = 0, \tag{2.5}$$

where

$$j^\mu = n\, u^\mu\,, \quad u^2 = -1\,, \tag{2.6}$$

n being the particle number density and u^μ the fluid four-velocity defined below. Moreover, if the charge flows with the fluid, charge conservation is obeyed separately by each volume element. This means that the charge conservation is not affected by an arbitrary comoving position-dependent transformation

$$\psi \to \psi + f(\phi^I) \tag{2.7}$$

f being an arbitrary function. This extra symmetry requirement on the Lagrangian is dubbed *chemical-shift symmetry.*

From these premises the authors of [6] construct the low energy Lagrangian respecting the above symmetries. At lowest order the Lagrangian will depend on the first derivatives of the fields through invariants respecting the symmetries (2.1)- (2.4) and (2.7) :

$$\mathcal{L} = \mathcal{L}(\partial\phi^I, \partial\psi). \tag{2.8}$$

For this purpose one introduces the following current which respects the symmetries (2.1)-(2.3):

$$J^\mu = \frac{1}{d!}\,\epsilon^{\mu,\nu_1,\ldots,\nu_d}\epsilon_{I_1,\ldots,I_d}\partial_{\nu_1}\phi^{I_1}\ldots\partial_{\nu_d}\phi^{I_d}\,, \tag{2.9}$$

and enjoys the important property that its projection along the comoving coordinates does not change:

$$J^\mu\,\partial_\mu\phi^I = 0. \tag{2.10}$$

This is equivalent to saying that the spatial d-form current $J^{(d)} = -\star_{(d+1)} J^{(1)}$, where

$$J^{(1)} = \frac{1}{d!}\,\epsilon_{\mu\nu_1\ldots\nu_d}\,\epsilon_{I_1\ldots I_d}\partial^{\nu_1}\,\phi^{I_1}\ldots\partial^{\nu_d}\,\phi^{I_d}\,dx^\mu = (-1)^d\,\star_{(d+1)}\left(\frac{1}{d!}\,\epsilon_{I_1\ldots I_d}d\phi^{I_1}\wedge\ldots\wedge d\phi^{I_d}\right)\,, \tag{2.11}$$

is closed identically, that is it is locally an exact form. Hence it is natural to define the fluid four-velocity as aligned with J^μ:

$$J^\mu = b\,u^\mu \to b = \sqrt{-J^\mu\,J_\mu} = \sqrt{\det(B^{IJ})}\,, \tag{2.12}$$

where $B^{IJ} \equiv \partial_\mu \phi^I \partial^\mu \phi^J$. From a physical point of view, the property of J^μ to be identically closed identifies it as the entropy current of the perfect fluid, so that $b = s$, s being the entropy density. Using the entropy current J^μ one finds that, by virtue of eq. (2.10), the quantity $J^\mu \partial_\mu \psi$ is invariant under (2.7).

Summarizing, a low energy Lagrangian invariant under (2.1)- (2.4) and (2.7), can depend on ϕ^I and ψ only through J^μ and $J^\mu \partial_\mu \psi$, and, being a Poincarè invariant, it can be written as follows:

$$S = \int d^4 x F(b, y) \,, \tag{2.13}$$

where y is

$$y = u^\mu \partial_\mu \psi = \frac{J^\mu \, \partial_\mu \psi}{b} \,. \tag{2.14}$$

Computing the Noether current associated with the symmetry (2.4) one derives

$$j^\mu = F_y \, u^\mu \to F_y \equiv n, \tag{2.15}$$

which identifies n as the particle number density. The Noether currents associated with the infinite symmetry (2.7) are

$$j^\mu_{(f)} = F_y \, u^\mu f(\phi^I), \tag{2.16}$$

and these currents are also conserved by virtue of the j^μ-conservation.

By coupling (2.13) to worldvolume gravity we can obtain the energy-momentum tensor by taking, as usual, the derivative with respect to the metric:

$$T_{\mu\nu} = (y \, F_y - b \, F_b) \, u_\mu \, u_\nu + \eta_{\mu\nu} \, (F - b \, F_b) \ . \tag{2.17}$$

On the other hand, from classical fluid-dynamics, we also have

$$T_{\mu\nu} = (p + \rho) \, u_\mu \, u_\nu + \eta_{\mu\nu} p \,, \tag{2.18}$$

from which we identify the pressure and density

$$\rho = y \, F_y - F \equiv y \, n - F \ \ , \ \ p = F - b \, F_b \,. \tag{2.19}$$

From the derivation of the energy-momentum tensor one can easily obtain the entropy density, the temperature, the Maxwell equations and so on (see [6] for a complete review). In particular, it turns out that the quantity y defined in eq. (2.14) coincides with the chemical potential μ. To see this, it suffices to compare the first principle

$$p + \rho = T \, s + \mu \, n \,, \tag{2.20}$$

with (2.19). Using $F_y = n$ and $b = s$, we then find

$$\frac{\partial F}{\partial s} = -T \ , \ \ y = \mu \,. \tag{2.21}$$

We conclude that the Lagrangian density is a function of s and μ

$$F = F(s, \mu) \,. \tag{2.22}$$

Let us remark that the Lagrangian used in this setting does not allow at first sight for the presence of a kinetic term for the dynamical field ψ, namely $X = \partial_\mu \psi \partial^\mu \psi$. In

fact we could also consider, besides X, further Poincaré invariants of the form $Z^I = \partial_\mu\psi\partial^\mu\phi^I$. However, it can be proven that the quantities X, Z^I, together with B^{IJ}, are not independent of y since the following relation holds:

$$y^2 = -\partial_\mu\psi\partial^\mu\psi + \partial_\mu\psi\partial^\mu\phi^I B^{-1}_{IJ}\partial_\nu\psi\partial^\nu\phi^J \,. \tag{2.23}$$

Therefore a dependence of the Lagrangian on X is somewhat implicit in y^2.

A Lagrangian exclusively depending on X, i.e. of the form $F(X)$, has been considered, for instance, in [8], to describe superfluids at $T = 0$. The use of the variables X, Z^I, even though redundant for ordinary fluids, can be useful in order to describe superfluids as a spontaneously broken phase of a field theory with chemical-shift symmetry invariance. We shall elaborate on this idea in a forthcoming note [19].

3 Supersymmetric Effective Theory in Two Space-Time Dimensions

3.1 Lagrangian and Equations of Motion

The Lagrangian formalism reviewed in the previous section will now be used as the starting point for a generalization of the Lagrangian approach to a supersymmetric theory. This could be a useful tool in the study of the fully supersymmetric fluid/gravity correspondence, but it is also an interesting point 'per se', which could be extended to local supersymmetry. As far as the entropy current is concerned, this last point will be developed in section 5. In order to avoid unnecessary cumbersome formulas, we first formulate the supersymmetrization in terms of fundamental fields $\phi(x,t)$ and $\psi(x,t)$ in two space-time dimensions, postponing its extension to (d+1) space-time dimensions to next section.

Let us consider the following superforms in 1+1 dimensional superspace (ϕ,θ) (using Nicolis et al. coordinates)

$$\Pi = d\phi + \theta\, d\theta \,, \qquad \Omega = d\psi + \tau\, d\theta \,, \tag{3.1}$$

satisfying:

$$d\Pi = d\theta \wedge d\theta \,, \qquad d(d\theta) = 0 \,, \qquad d\Omega = d\tau \wedge d\theta \,. \tag{3.2}$$

They are 1-forms and they are respectively invariant and covariant under the following transformations

$$\delta\phi = -\epsilon\theta \,, \quad \delta\theta = \epsilon \,, \quad \delta\psi = f(\phi,\theta) \,, \quad \delta\tau = -Df(\phi,\theta) \,, \tag{3.3}$$

where $D = \partial_\theta - \theta\partial_\phi$ with $D^2 = -\partial_\phi$. The transformation of Π and Ω under the second set of transformations is given by

$$\delta\Pi = 0 \,, \qquad \delta\Omega = \Pi\, \partial_\phi f \,. \tag{3.4}$$

The first transformation is a supersymmetry transformation of the comoving variables (notice that since ϕ, or in general ϕ^I are the comoving coordinates of the fluid, we associate with them a set of anticommuting coordinates θ, or in general θ^α – see also [11]) while the second transformation is the chemical shift symmetry introduced in [6].

In terms of these variables, we can build the following quantities

$$B = -\Pi \wedge {}^\star\Pi = \Pi_\mu g^{\mu\nu}\Pi_\nu d^2x = \hat{B}\, d^2x\,,$$
$$Y = \Omega \wedge \Pi = \epsilon^{\mu\nu}\Omega_\mu \Pi_\nu d^2x = \hat{Y}\, d^2x\,. \tag{3.5}$$

We denote by the same letter, but with a hat on the top, the corresponding quantity modulo the volume form, that is $\hat{B} = \Pi_\mu \eta^{\mu\nu}\Pi_\nu$, $\hat{Y} = \Omega_\mu \epsilon^{\mu\nu}\Pi_\nu$, $b = \sqrt{\hat{B}}$.

They are invariant under supersymmetry and chemical-shift symmetry.

The variation of the Poincaré-invariant superfields under a generic variation of ϕ and of ψ is

$$\delta B = -2\Pi \wedge {}^\star d\delta\phi\,,$$
$$\delta Y = -\Pi \wedge d\delta\psi + \Omega \wedge d\delta\phi\,, \tag{3.6}$$

Therefore, if the action is given by

$$S = \int d^2x F[b, Y] \tag{3.7}$$

as an integral of a local functional, we get the equations of motion

$$d\Big[b\, F_b{}^\star\Pi + F_Y\Omega\Big] = d^\star J_\phi = 0\,, \tag{3.8}$$

$$d\Big[F_Y\Pi\Big] = d^\star j = 0\,, \tag{3.9}$$

where $F_b = \partial F/\partial b\,, F_Y = \partial F/\partial Y$, and we have defined the 1-forms:

$$J_\phi = b\, F_b\Pi + F_Y{}^\star\Omega\;;\;\; j = F_Y{}^\star\Pi\,, \tag{3.10}$$

which are the Noether currents associated with the shift symmetries $\phi \to \phi + c'$, $\psi \to \psi + c$. The variations for θ and τ are

$$\delta B = -2\Pi \wedge {}^\star(\delta\theta d\theta + \theta d\delta\theta)\,,$$
$$\delta Y = -\Pi \wedge (\delta\tau d\theta + \tau d\delta\theta) + \Omega \wedge (\delta\theta d\theta + \theta d\delta\theta)\,, \tag{3.11}$$

and the corresponding equations of motion are

$$2^\star J_\phi \wedge d\theta + d\tau \wedge {}^\star j = 0\,,$$
$${}^\star j \wedge d\theta = 0\,, \tag{3.12}$$

We can now compute the supercurrent and the current associated with the chemical shift symmetry. It is easy to show that the first current corresponds to

$$j_S = -2\,\theta\,{}^\star J_\phi + \tau\,{}^\star j\,, \tag{3.13}$$

which, by the equations of motion, enjoys the property: $d^\star j_S = 0$.

Concerning the chemical shift symmetry, we first consider the possibility of constant symmetry, namely $f = a + \omega\theta$ where a and ω are commmunting and anticommuting parameters of constant type, respectively. Therefore, it is easy by Noether method to compute the following two currents

$$J^{(a)} = j\,, \qquad J^{(\omega)} = j\,\theta\,, \tag{3.14}$$

which can be obviously cast into a supermultiplet. Notice that $d^\star J^{(a)} = 0$ as follows from the second equation of (3.8), while $d^\star J^{(\omega)} = 0$ follows from the second equation of (3.12).

3.2 Entropy Current

We use the formalism of the previous section to formulate the entropy current. The Lagrangian formulation is suitable for the present derivation. We start in this section with the 1+1 case and then we extended it to $d+1$ case in Section 4.

We are finally able to define the entropy current with the following properties

1. It is a one-form (in the two dimensional case, otherwise it is a d -form for $d+1$ dimensional space).
2. It is supersymmetric (invariant under the transformations (3.3)).
3. In a suitable limit it reduces to the bosonic expression (see for example [6]).
4. It is conserved off-shell.
5. Its Hodge-dual is conserved on-shell. [3]

The result is the following expression

$$\boxed{J = \Pi \wedge \delta(d\theta)} \tag{3.15}$$

where $\delta(d\theta)$ is the Dirac delta function of the differential $d\theta$. Notice that $d\theta \wedge d\theta \neq 0$ since $d\theta$ is a commuting quantity. Its Hodge-dual on the supermanifold is

$$^{\star}J = {}^{\star}\Pi\theta \tag{3.16}$$

as explained in Appendix A.[4]

The properties of $\delta(d\theta)$ are discussed in [17] and summarized in Appendix A2. We just recall the most important features useful to show the properties of the entropy current in the present two dimensional case and in the $(d+1)$-dimensional case discussed in Section 4.

First of all $\delta(d\theta^\alpha)$ enjoys the usual equation $d\theta^\alpha\, \delta(d\theta^\alpha) = 0$.[5] In addition, $\delta(d\theta^\alpha)$ carries no form-degree and therefore the current $J^{(d|m)}$ is a d-form. However, we can assign a new quantum number m, dubbed *picture number*, which takes into account the number of the Dirac delta-functions $\delta(d\theta^\alpha)$. Thus a p-form of picture q is denoted $\omega^{(p|q)}$. Notice that by using the properties of the Dirac delta-functions it is easy to show that $\delta(d\theta^\alpha)_\wedge \delta(d\theta^\beta) = -\delta(d\theta^\beta)_\wedge \delta(d\theta^\alpha)$ and therefore any integral in superspace of a p-form ($p \leqq d+1$) with a given picture-number q cannot have more than one delta-function of a given differential $d\theta^\alpha$.

Furthermore one can also consider derivatives of Dirac delta-functions which have degree $(-n|1)$ where the first label corresponds to n-th derivative of the Dirac-delta function and whose properties are given in Appendix A2.

With the definitions introduced above, we can write J in (3.15) as $J^{(1|1)}$. Its 0-picture component, evaluated at $\theta = 0$, reduces to the bosonic expression $J_{bos} = d\phi$. Furthermore,

[3] Note that this last property only holds in (1+1) dimensions.

[4] For earlier definitions of Hodge duality on supermanifolds, though in a different context, see for instance [20].

[5] The index $\alpha = 1, \cdots, m$ enumerates the components of the spinor representation. For the present $d = 1$ case $m = 1$ and we can omit the index alpha.

we have to notice that $J = \Pi \wedge \delta(d\theta) = d\phi \wedge \delta(d\theta)$. The former expression is manifestly supersymmetric. To check its properties, we observe

$$dJ = d\Pi \wedge \delta(d\theta) = d\theta \wedge d\theta \wedge \delta(d\theta) = 0\,. \tag{3.17}$$

We have to notice that in the case of the bosonic entropy current $J = d\phi$, we can construct an infinite number of currents by multiplying it by any function $f(\phi)$, since then $J_f = f(\phi)d\phi$ is clearly closed. Let us check if this is also possible in the case of the fermionic generalization. An infinite set of currents can be defined also in the supersymmetric version by setting $f(\phi, \theta)\Pi \wedge \delta(d\theta)$. However, in the supersymmetric case there is another possibility.
To understand this point, it is interesting to observe that the expression

$$\eta_{(0|1)} = (\theta\, \delta(d\theta) + \Pi \wedge \delta'(d\theta)) \tag{3.18}$$

is a (0|1)-form since the first term is a pure Dirac delta function (which carries no form degree) and the second term is made of a 1-form, namely Π, and a (-1)-form, namely $\delta'(d\theta)$. Acting on $\eta_{(0|1)}$ with the differential d, we have

$$\begin{aligned} d\eta_{(0|1)} &= d\Big(\theta\, \delta(d\theta) + \Pi \wedge \delta'(d\theta)\Big) = d\Pi \wedge \delta'(d\theta) \\ &= d\theta \wedge d\theta \wedge \delta'(d\theta) = -d\theta \wedge \delta(d\theta) = 0 \end{aligned} \tag{3.19}$$

In addition, we can define a new current

$$\eta_{(-1|1)} = (\theta\, \delta'(d\theta) + \Pi \wedge \delta''(d\theta))\ . \tag{3.20}$$

where $\delta''(d\theta)$ is the second derivative of the Dirac delta function. The quantity $\eta_{(-1|1)}$ is a (-1) form because the derivatives on Dirac delta functions count as negative form number. Again, $d\eta_{(-1|1)} = 0$ using the properties of delta functions. Finally, we can define an infinite set of currents of the form

$$\eta_{(-n|1)} = \big(\theta\, \delta^{(n)}(d\theta) + \Pi \wedge \delta^{(n+1)}(d\theta)\big)\ , \tag{3.21}$$

satisfying $d\eta_{(-n|1)} = 0$.

We note that the current $J = d\phi \wedge \delta(d\theta)$ can be written as an exact form $J = d\left(\phi\delta(d\theta)\right)$. However, this expression (as also in the purely bosonic case) is an exact expression, but $\phi\delta(d\theta)$ is not globally defined because of the uncovered coordinate ϕ. Nonetheless,in the supersymmetric case another possibility is

$$J = d\left(d\phi \wedge \theta\delta'(d\theta)\right)\ , \tag{3.22}$$

which is globally defined (it is not clear what is an anti-commuting non globally defined variable).[6] Nonetheless, the expression $d\phi \wedge \theta\delta'(d\theta)$ is not supersymmetric and therefore J cannot be consider as an exact expression.

[6]A deeper discussion on this point is published in [17] where global vs. non-global issues are discussed using the Čech cohomology for projective superspaces.

3.3 Generic Expression for the Entropy Current

In the present section, we consider a generalised form of the entropy current and the relation between the different forms. This is related to the fact that the integral forms can be seen also from a gauge fixing point of view.[7]

In our case, for the 1+1 dimensional case we can consider the following expressions

$$P = d\Phi + \Theta d\Theta \,, \quad \delta(d\Theta)\,, \quad d\Theta\,. \tag{3.23}$$

where Φ and Θ are functions of the coordinates ϕ, θ. Therefore the generalised expression becomes

$$J^{(1|1)} = P_\wedge \delta(d\Theta)\,. \tag{3.24}$$

It is easy to connect it to the original formula (3.15) by expressing (3.23) in terms of the coordinates. This can be easily done by observing

$$\begin{aligned} J^{(1|1)} &= \Big[d\phi\Big(\partial_\phi\Phi + \Theta\partial_\phi\Theta\Big) + d\theta\Big(\partial_\theta\Phi - \Theta\partial_\theta\Theta\Big)\Big]\,\delta\Big(d\theta\partial_\theta\Theta + d\phi\partial_\phi\Theta\Big) = \\ &= \Big[d\phi\Big(\partial_\phi\Phi + \Theta\partial_\phi\Theta\Big) + d\theta\Big(\partial_\theta\Phi - \Theta\partial_\theta\Theta\Big)\Big]\,\frac{1}{\partial_\theta\Theta}\left[\delta\Big(d\theta + d\phi\frac{1}{\partial_\theta\Theta}\partial_\phi\Theta\Big)\right] = \\ &= \Big[d\phi\Big(\partial_\phi\Phi + \Theta\partial_\phi\Theta\Big) + d\theta\Big(\partial_\theta\Phi - \Theta\partial_\theta\Theta\Big)\Big]\,\frac{1}{\partial_\theta\Theta}\left[\delta(d\theta) + d\phi\frac{1}{\partial_\theta\Theta}\partial_\phi\Theta\,\delta'(d\theta)\right] = \\ &= d\phi_\wedge\delta(d\theta)\left(\frac{\partial_\phi\Phi - \partial_\phi\Theta\partial_\theta\Phi(\partial_\theta\Theta)^{-1}}{\partial_\theta\Theta}\right) = \\ &= d\phi_\wedge\delta(d\theta)\,\mathrm{sdet}\mathcal{J}\,. \end{aligned} \tag{3.25}$$

The expression sdet$\mathcal{J}$ is the super determinant of the super matrix $\mathcal{J} = \begin{pmatrix} \partial_\phi\Phi & \partial_\theta\Phi \\ \partial_\phi\Theta & \partial_\theta\Theta \end{pmatrix}$. This proves that the relation between the two formulas is simply the super determinant of the Jacobian matrix.

The generalized expression (3.24) could be the appropriate form in order to study the supersymmetric entropy current in a fluid/gravity context (see for example [1],[13],[21]). In the bosonic case the entropy density is proportional to the black-hole horizon area. Then the supersymmetric version of the area increase theorem can be expressed as the statement

$$\frac{\partial}{\partial\lambda}\mathrm{sdet}\mathcal{J} \geq 0 \tag{3.26}$$

where λ is the additional bosonic coordinate orthogonal to the super surface whose volume element is the $(1|1)$ integral form (3.23). The present derivation can be extended to the $d+1$ dimensional case and results in the substitution of the matrix sdet$\mathcal{J}$ with the corresponding super matrix in the appropriate dimensions.

[7] This was the original point of view for introducing the PCO (Picture Changing Operator) which are written in terms of integral form in superstring formulation. To make a long story short, we recall that in the case of superstring the gauge symmetry is a local symmetry plus worldsheet diffeomorphisms and therefore its quantization proceeds by fixing those symmetries by a gauge-fixing-BRST methods. In that process, we have to choose a background metric and a background gravitino. For example, one simple choice is to set the gravitino to zero. However, the corresponding ghost – needed to implement the BRST formalism for that gauge summetry – is a commuting ghost (usually denoted by β) and the functional integral on it yields the Dirac delta function for the gravitino. Obviously, one can choose a different gauge fixing. [22, 23, 24].

Let us check how the supersymmetry transformations act on this generalized expression for the entropy current. Denoting by $\delta_\epsilon \Theta$ the supersymmetry transformation of Θ, we have

$$\delta_\epsilon J^{(1|1)} = d\Big(J^{(0|1)}\delta_\epsilon\Theta\Big)\,. \tag{3.27}$$

where $dJ^{(0|1)} = 0$. It is not invariant but, as discussed above, the entropy density is however invariant since it is integrated on the surface. In the case where Φ and Θ do coincide with the original coordinates the right hand side of (3.27) vanishes since $\delta_\epsilon\Theta = \delta_\epsilon\theta = \epsilon$ is constant.

4 The General $d+1$ Dimensional Case

In this section the construction of the previous section will be generalized to $(d+1)$-space-time dimensions.

Let us consider the following 1-forms in superspace

$$\Pi^I = d\phi^I + \frac{i}{2}\bar\theta\Gamma^I\, d\theta\,, \qquad \Omega = d\psi + i\bar\tau\, d\theta\,. \tag{4.1}$$

Here Γ^I are the Clifford algebra Γ-matrices in $d+1$-dimensions, with the index $I = 1,\dots,d$ running on the spatial components only of Γ^a, $a = 0,1,\cdots,d$, while θ, τ and $d\theta$ denote the matrix form of the *Majorana* spinors in the $m = 2^{d/2}$-dimensional spinor representation of $\mathrm{SO}(d,1)$. [8] Notice that since ϕ^I are the comoving coordinates of the fluid, we associate with them a set of anticommuting coordinates θ^α, $\alpha = 1,\dots,m$, see also [11]).
Π^I and Ω 1-forms are invariant under the following supersymmetry transformations of the fundamental fields ϕ^I,θ^α:

$$\delta\phi^I = -i\bar\epsilon\Gamma^I\theta\,, \quad \delta\theta = \epsilon \tag{4.2}$$

One can also generalize the chemical-shift symmetry of the bosonic case by setting

$$\delta\psi = f(\phi,\theta), \qquad \delta\Omega = \Pi^I\,\partial_{\phi^I} f\,. \tag{4.3}$$

while Π^I remains invariant. Equation (4.3) also implies

$$\delta\tau_\alpha = -D_\alpha f(\phi,\theta)\,,$$

where $D_\alpha = \partial_{\theta^\alpha} - \frac{i}{2}\bar\theta^\beta\Gamma^I_{\beta\alpha}\partial_{\phi^I}$.

Note that under d-differentiation we have

$$d\Pi^I = \frac{i}{2}d\bar\theta\wedge\Gamma^I d\theta\,, \qquad d(d\theta) = 0\,, \qquad d\Omega = d\bar\tau\wedge d\theta\,. \tag{4.4}$$

In terms of the above variables, we can build the following quantities

$$\begin{aligned} B^{IJ} &= -\Pi^I\wedge{}^\star\Pi^J = \Pi^I_\mu\Pi^{J\mu}\,d^{d+1}x = \hat B^{IJ}\,d^{d+1}x\,,\\ Y &= \Omega\wedge\Pi^1\cdots\wedge\Pi^d = \frac{1}{d!}\epsilon^{\mu\nu_1\cdots\nu_d}\epsilon^{I_1\cdots I_d}\Omega_\mu\Pi^{I_1}_{\nu_1}\cdots\Pi^{I_d}_{\nu_d}\;d^{d+1}x = \hat Y\,d^{d+1}x\,, \end{aligned} \tag{4.5}$$

[8] For *Majorana-Weyl* spinors, the dimension of the representation is instead $m = 2^{(d-1)/2}$.

where we have denoted by the same letter, though with a hat on the top, the corresponding factor multiplying the volume form. Note in particular that [6]:

$$\hat{b} = \sqrt{\det \hat{B}^{IJ}}\,, \tag{4.6}$$

and

$$\hat{b}\Big|_{\theta=0} = s\ ;\quad \frac{\hat{Y}}{\hat{b}}\Bigg|_{\theta=0} = y\,. \tag{4.7}$$

The quantities $\hat{b}$ and $\hat{Y}$ are manifestly invariant under supersymmetry transformations and chemical-shift symmetry.

On the basis of the previous discussion the action generalizing (3.7) will be written as the following integral of a local functional

$$S = \int d^{d+1}x F[\hat{b},\hat{Y}]. \tag{4.8}$$

The variation of the Poincaré-invariant superfields under a generic variation of ϕ^I and of ψ is

$$\begin{aligned}
\delta B^{IJ} &= -2\Pi^I \wedge {}^\star d\delta\phi^J\,,\\
\delta Y &= d\delta\psi \wedge \Pi^1 \wedge \cdots \wedge \Pi^d + \frac{1}{(d-1)!}\epsilon^{I_1\cdots I_d}\Omega \wedge \Pi^{I_1} \wedge \cdots \wedge \Pi^{I_{d-1}} \wedge d\delta\phi^{I_d}\,,
\end{aligned} \tag{4.9}$$

and the following equations of motion are obtained:

$$d^\star J_I^{(1)} = d\Big[\hat{b}\,F_b \hat{B}^{-1\star}_{IJ}\Pi^J + \frac{1}{(d-1)!}\epsilon^{IJ_1\cdots J_{d-1}}F_Y\Omega \wedge \Pi^{J_1} \wedge \cdots \wedge \Pi^{J_{d-1}}\,, \tag{4.10}$$

$$d^\star j^{(1)} = d\Big[F_Y \Pi^1 \wedge \cdots \wedge \Pi^d\Big] = 0\,, \tag{4.11}$$

where $F_b = \partial F/\partial\hat{b}$, $F_Y = \partial F/\partial\hat{Y}$ and we have introduced the two currents:

$$J_I^{(1)} = \hat{b}\,F_b \hat{B}^{-1}_{IJ}\Pi^J + \frac{1}{(d-1)!}\Omega_\mu \epsilon^{\mu\nu\mu_1\ldots\mu_{d-1}}\epsilon^{IJ_1\cdots J_{d-1}}F_Y \Pi^{J_1}_{\mu_1}\cdots\Pi^{J_{d-1}}_{\mu_{d-1}}dx_\nu\,, \tag{4.12}$$

$$j^{(1)} = -F_Y\,\epsilon^{\mu\mu_1\ldots\mu_d}dx_\mu \Pi^1_{\mu_1}\ldots\Pi^1_{\mu_d}\,. \tag{4.13}$$

It is straightforward to verify that the above quantities are the Noether currents associated with the constant translational symmetries $\phi^I \to \phi^I + c^I$ and $\psi \to \psi + c$. Furthermore, a general variation of the fermionic fields θ and τ, gives

$$\begin{aligned}
\delta B^{IJ} &= -i\Pi^I \wedge {}^\star(\delta\bar{\theta}\Gamma^J d\theta + \bar{\theta}\Gamma^J d\delta\theta)\,,\\
\delta Y &= i\,(\delta\bar{\tau}d\theta + \bar{\tau}d\delta\theta) \wedge J^{(d)} + \frac{i}{2}\frac{1}{(d-1)!}\epsilon^{I_1\cdots I_d}\Omega \wedge \Pi^{I_1} \wedge \cdots \wedge \Pi^{I_{d-1}} \wedge (\delta\bar{\theta}\Gamma^{I_d}d\theta + \bar{\theta}\Gamma^{I_d}d\delta\theta)\,,
\end{aligned} \tag{4.14}$$

so that the corresponding *fermionic* equations of motion are

$$J_I^{(1)\mu}\,\Gamma^I\partial_\mu\theta + \eta_C\,j^\mu\,\partial_\mu\tau = 0\,, \tag{4.15}$$

where η_C is the sign appearing in the relation $\bar{\tau}d\theta = \eta_C\,\bar{d\theta}\tau$ and depends on the property of the charge-conjugation matrix in $(d+1)$-dimensions.

4.1 Entropy Current

We now use the general setting introduced in the previous section to formulate the entropy current.

Our formalism allows us to define the supersymmetric extension of the purely bosonic entropy current (2.9) with the following properties:

1. It is a purely spatial d-form on space-time;
2. It is conserved off-shell;
3. Its zero-picture part, at $\theta = 0$, reduces to the bosonic expression in ordinary hydrodynamics;
4. It is supersymmetric (invariant under the transformations (4.2)),

The result is the following expression

$$\boxed{J^{(d|m)} = \frac{1}{d!}\epsilon_{I_1 \ldots I_d} \Pi^{I_1} \wedge \cdots \wedge \Pi^{I_d} \bigwedge_{\alpha=1}^{m} \delta(d\theta^\alpha)} \tag{4.16}$$

where $\delta(d\theta^\alpha)$ is the Dirac delta function of the differential $d\theta^\alpha$ (recall that $d\theta^\alpha$ is a commuting quantity) and m is the dimension of the spinor representation in $d+1$-dimensions.

It is now easy to show that $J^{(d|m)}$ is off-shell closed. Indeed:

$$\begin{aligned} dJ^{(d|m)} &= \frac{1}{(d-1)!}\, \epsilon_{I_1 I_2, \ldots I_d}\, d\Pi^{I_1} \wedge \Pi^{I_2} \cdots \wedge \Pi^{I_d} \bigwedge_{\beta=1}^{m} \delta(d\theta^\beta) + \\ &+ \frac{(-1)^d\, m}{d!}\, \epsilon_{I_1 I_2, \ldots I_d}\, \Pi^{I_1} \wedge \Pi^{I_2} \cdots \wedge \Pi^{I_d} \wedge \left[\sum_{\beta=1}^{m} \delta'(\delta\theta^\beta) \wedge d^2\theta^\beta \bigwedge_{\alpha \neq \beta} \delta(\Psi^\alpha) \right] . \end{aligned} \tag{4.17}$$

The first term is zero since from

$$d\Pi^{I_i} = \frac{\mathrm{i}}{2} d\overline{\theta} \wedge \Gamma^{I_i} d\theta \,,$$

any component of the spinor $d\theta$ in the current $d\overline{\theta} \wedge \Gamma^{I_i} d\theta$ hitting a corresponding $\delta(d\theta)$ gives zero, while the second term vanishes trivially because $d\, d\theta^\alpha \equiv 0$. Therefore

$$dJ^{(d|m)} = 0 \,. \tag{4.18}$$

Moreover the 0- picture part $J^{(d|0)}$ of $J^{(d|m)}$ computed at $\theta = 0$ obviously coincides with the entropy current of the ordinary bosonic hydrodynamics

$$J^{(d|0)}_{|\theta=0} = J^{(d)} \,.$$

Finally, recalling that both Π^I and $d\theta^\alpha$ are invariant under supersymmetry, we conclude that $J^{(d|m)}$ is a supersymmetric invariant. Actually, as in the bosonic case, there is an infinity of d-form currents which are off-shell closed (and therefore their Hodge-dual are conserved). Indeed if we define

$$J^{(d|m)}_f = f(\phi^I, \theta)\, J^{(d|m)} \,, \tag{4.19}$$

then

$$d\left[f(\phi^I,\theta)\,J^{(d|m)}\right]=\left[\frac{\partial f}{\partial\phi^J}d\phi^J+\frac{\partial f}{\partial\theta^\alpha}d\theta^\alpha\right]\,J^{(d|m)}=0 \tag{4.20}$$

An analogous formula will be discussed in the case of supergravity (see Section 5).

Finally, we show that we can define an infinite set of fermionic closed $(-n|1)$-superforms analogous to those defined in the two-dimensional case (see eq.s (3.19), (3.20), (3.21)). Let us introduce the following fermionic $(-n|1)$-superforms:

$$J^\alpha_{\beta_1\dots\beta_n}=\mathbb{P}^{\delta_1\dots\delta_n}_{\beta_1\dots\beta_n}\,\hat{J}^\alpha_{\delta_1\dots\delta_n}\,, \tag{4.21}$$

where $\mathbb{P}$ is the projector onto the *irreducible* n-fold symmetric product of the spinorial representation and

$$\begin{aligned}\hat{J}^\alpha_{\beta_1\dots\beta_n}&\equiv\theta^\alpha\,\partial_{\beta_1}\cdots\partial_{\beta_n}\prod_\beta\delta(d\theta^\beta)+\frac{i}{d}\Pi_I\,(\Gamma^I C^{-1})^{\alpha\gamma}\partial_\gamma\partial_{\beta_1}\cdots\partial_{\beta_n}\prod_\beta\delta(d\theta^\beta)\,,\\ \partial_\beta&\equiv\frac{\partial}{\partial d\theta^\beta}\,,\end{aligned} \tag{4.22}$$

where C is the charge-conjugation matrix. To prove it we need to use the property:

$$\begin{aligned}d\theta^\alpha\,\partial_{\beta_1}\cdots\partial_{\beta_k}\prod_\beta\delta(d\theta^\beta)&=k\,\delta^\alpha_{(\beta_1}\partial_{\beta_2}\cdots\partial_{\beta_k)}\prod_\beta\delta(d\theta^\beta)\,,\\ d\theta^\alpha\,d\theta^\gamma\partial_{\beta_1}\cdots\partial_{\beta_k}\prod_\beta\delta(d\theta^\beta)&=k(k-1)\,\delta^{(\alpha\gamma)}_{(\beta_1\beta_2}\partial_{\beta_3}\cdots\partial_{\beta_k)}\prod_\beta\delta(d\theta^\beta)\,.\end{aligned} \tag{4.23}$$

Exterior derivation of (4.22) then yields:

$$\begin{aligned}d\hat{J}^\alpha_{\beta_1\dots\beta_n}=&\,n\,\delta^\alpha_{(\beta_1}\partial_{\beta_2}\cdots\partial_{\beta_n)}\prod_\beta\delta(d\theta^\beta)+\frac{i}{d}\frac{i}{2}\,d\theta^T C\Gamma_I d\theta\,(\Gamma^I C^{-1})^{\alpha\gamma}\partial_\gamma\partial_{\beta_1}\cdots\partial_{\beta_n}\prod_\beta\delta(d\theta^\beta)=\\ =&\,n\,\delta^\alpha_{(\beta_1}\partial_{\beta_2}\cdots\partial_{\beta_n)}\prod_\beta\delta(d\theta^\beta)-\frac{n(n+1)}{2d}\,(\Gamma^I C^{-1})^{\alpha\gamma}\,(C\Gamma_I)_{(\gamma\beta_1}\,\partial_{\beta_2}\cdots\partial_{\beta_n)}\prod_\beta\delta(d\theta^\beta)\\ =&\,n\,\delta^\alpha_{(\beta_1}\partial_{\beta_2}\cdots\partial_{\beta_n)}\prod_\beta\delta(d\theta^\beta)-\\ &-\frac{n}{2d}\,(\Gamma^I C^{-1})^{\alpha\gamma}\,\left(2(C\Gamma_I)_{\gamma(\beta_1}\,\partial_{\beta_2}\cdots\partial_{\beta_n)}+(n-1)\,(C\Gamma_I)_{(\beta_1\beta_2}\,\partial_{\beta_3}\cdots\partial_{\beta_n)}\partial_\gamma\right)\prod_\beta\delta(d\theta^\beta)=\\ =&-\frac{n(n-1)}{2d}\,(\Gamma^I C^{-1})^{\alpha\gamma}\,(C\Gamma_I)_{(\beta_1\beta_2}\,\partial_{\beta_3}\cdots\partial_{\beta_n)}\partial_\gamma\prod_\beta\delta(d\theta^\beta)\,,\end{aligned} \tag{4.24}$$

the latter term vanishes upon contraction with $\mathbb{P}$.

4.2 Entropy Density

For a generic real supermanifold $\mathcal{M}^{(n+1|m)}$, the Hodge dual (see Appendix A3) is defined as follows

$$(\omega^{(p|q)},\eta^{(p|q)})=\int_{\mathcal{M}^{(d+1|m)}}\omega^{(p|q)}\wedge{}^\star\eta^{(p|q)}\,, \tag{4.25}$$

where $\omega^{(p|q)}$ is a p-form with q delta functions. Notice that for each Grassmann coordinate we have $\int \theta^\alpha \delta(d\theta^\alpha) = 1$, see eq. (A.13), where the integral has to be understood as a Berezin integral over θ^α and as distributional integral (namely acting on a test function $g(d\theta^\alpha)$) over the coordinate $d\theta^\alpha$. If we want to compute the entropy density associated with the entropy supercurrent we can make use of the formula (4.25). For example, for the two-dimensional case we have

$$(J^{(1|1)}, J^{(1|1)}) = \int_{\mathcal{M}^{(2|1)}} \Pi \wedge \delta(d\theta) \wedge (^\star \Pi \theta) = \int_{\mathcal{M}^{(2|1)}} \Pi \wedge {}^\star \Pi \wedge \theta \delta(d\theta) = -\int_{\mathcal{M}^{(2)}} \partial_\mu \phi \, \partial^\mu \phi \, d^2 x \, . \tag{4.26}$$

so that the two-dimensional entropy density is $\sqrt{\partial_\mu \phi \, \partial^\mu \phi}$.

In general, we can construct a formula for $(d+1|m)$ dimensional superspace :

$$J^{(d+1|m)} = \frac{1}{d!} \epsilon_{I_1 \ldots I_n} \Pi^{I_1} \wedge \cdots \wedge \Pi^{I_d} \wedge \delta(d\theta^1) \wedge \cdots \wedge \delta(d\theta^m) \, . \tag{4.27}$$

In that case we can compute the entropy density as follows:

$$\begin{aligned}
(J^{(d|m)}, J^{(d|m)}) &= \int_{\mathcal{M}^{(d+1|m)}} J^{(d|m)} \wedge {}^\star J^{(d|m)} \\
&= \int_{\mathcal{M}^{(d+1|m)}} \bigwedge_{I=1}^{d} \Pi^I \wedge {}^\star \bigwedge_{J=1}^{d} \Pi^J \prod_{\alpha=1}^{m} \theta^\alpha \delta(d\theta^\alpha) \\
&= -\int_{\mathcal{M}^{(d+1|m)}} \det \left(\Pi^I_\mu \Pi^{J,\mu} \right) Vol_{d+1} \prod_{\alpha=1}^{m} \theta^\alpha \delta(d\theta^\alpha) \, ,
\end{aligned} \tag{4.28}$$

where Vol_{d+1} is the volume element of the bosonic worldvolume. This confirms that the expression $\hat{b} = \sqrt{\det \left(\Pi^I_\mu \Pi^{J,\mu} \right)}$ represents the generalization for a $d+1$ dimensional fluid of the entropy density. The main difference is that it has been built with supersymmetric invariant building blocks and therefore it enjoys the invariance under supersymmetry transformations.

4.3 The Energy-Momentum Tensor

In equation (4.6) we have seen that the generalization of the entropy density $b \equiv s$

$$s(x) = s(\partial_\mu \phi^I) = \sqrt{\det \partial_\mu \phi^I \, \partial^\mu \phi^J} \tag{4.29}$$

to a supersymmetric setting is simply obtained by replacing the purely spatial rigid vielbein $d\phi^I$ with its supersymmetric version $\Pi^I(x, \theta) = d\phi^I + \frac{\mathrm{i}}{2} \overline{\theta} \Gamma^I d\theta$, so that the entropy density superfield is given by

$$\hat{s}(x) = \hat{s} \left(\partial_\mu \phi^I(x) \, , \partial_\mu \theta^\alpha(x) \right) = \sqrt{\det \Pi^I_\mu \, \Pi^{J\mu}}. \tag{4.30}$$

In an analogous way we generalize the bosonic variable of Section 2, $y = (J^\mu / s) \, \partial_\mu \psi$, to

$$\hat{y} = \frac{J^{\mu(1|0)}}{\hat{s}} \, \Omega_\mu = \frac{\hat{Y}}{\hat{s}} \, , \tag{4.31}$$

where $J^{(1|0)\mu}$ is defined by the Hodge-dual of the zero picture part of equation (4.18), namely

$$J^{(1|0)\mu} = \frac{1}{d!} \epsilon^{\mu \nu_1 \cdots \nu_d} \epsilon_{I_1 \ldots I_d} \Pi^{I_1}_{\nu_1} \ldots \Pi^{I_d}_{\nu_d} \, . \tag{4.32}$$

The generalization of the energy-momentum tensor of the supersymmetric theory is then obtained by varying the action functional of the superfields, S, given by the d-dimensional generalization of (4.8):

$$S = \int d^{d+1}\, x\sqrt{-g}\, F(\hat{s}, \hat{y})\,, \tag{4.33}$$

with respect to a probe metric $g_{\mu\nu}$. It is straightforward to see that we obtain for the energy-momentum tensor *formally* the same result as in the non-supersymmetric case, the only difference being the substitution of b, y with their supersymmetric counterparts $\hat{s}(x, \theta), \hat{y}(x, \theta)$:

$$T_{\mu\nu} = \left(\hat{y}\frac{\partial F}{\partial \hat{y}} - \hat{s}\frac{\partial F}{\partial \hat{s}}\right) u_\mu u_\nu + \eta_{\mu\nu}\left(F - \hat{s}\frac{\partial F}{\partial \hat{s}}\right)\,. \tag{4.34}$$

It follows that also the relation between the thermodynamic functions and the superfields $\hat{s}(x, \theta), \hat{y}(x, \theta)$ remains formally the same, namely

$$p \;=\; F(\hat{s}, \hat{y}) - \hat{s}(x, \theta)F_s\,, \tag{4.35}$$

$$\rho \;=\; \hat{y}(x, \theta)\, \hat{n} - F(\hat{s}, \hat{y})\,, \tag{4.36}$$

so that the pressure and the energy density become superfields whose $\theta = 0$ components give the usual field variables $\rho(x), p(x)$. The same of course happens for all the other thermodynamical variables.

5 Entropy Current for Supergravity

5.1 $\mathcal{N} = 1$ Supergravity

The above formulas, constructed for rigid supersymmetry, can be generalized to local supersymmetry, that is supergravity.

Let us work for simplicity in the case of $\mathcal{N} = 1$, D=4 supergravity coupled to a set of chiral multiplets (z^i, χ^i); $(z^{\bar{\imath}}, \chi^{\bar{\imath}})$ $(i, \bar{\imath} = 1, \cdots n_c)$. As we will discuss in Section 5.2, the result can be generalized to higher dimensional and/or extended supergravities. Let V^a be the (bosonic) vielbein ($a = 0, 1, 2, 3$ denote rigid space-time indices) and Ψ^α the gravitino (fermionic) one-form in superspace. The supertorsion T^a and the gravitino field-strength ρ 2-forms in superspace are defined as follows:

$$T^a \;=\; dV^a - \omega^a{}_b \wedge V^b - \frac{i}{2}\, \bar{\Psi}\gamma^a\Psi\,, \tag{5.1}$$

$$\rho \;\equiv\; \nabla\Psi = d\Psi - \frac{1}{4}\gamma^{ab}\omega_{ab}\Psi\,, \tag{5.2}$$

where ω^{ab} is the spin connection. Their on-shell parametrization in superspace (superspace constraints) is

$$T^a \;=\; 0 \tag{5.3}$$

$$\rho \;=\; \rho_{ab}V^a \wedge V^b + \mathrm{L}_a\, \gamma_5\, \gamma^{ab}\Psi \wedge V^b + \left[(\mathrm{Re}\, S) + \mathrm{i}\gamma^5(\mathrm{Im}\, S)\right] \gamma^a\, \Psi \wedge V_a\,, \tag{5.4}$$

where $\rho_{ab}V^a \wedge V^b$ is the supercovariant gravitino field strength, $\mathrm{L}_a = \frac{i}{8}\chi^i\gamma^{ab}\chi^{\bar{\imath}}g_{i\bar{\jmath}}$ is a current of spin-$\frac{1}{2}$ left-handed and right-handed chiral fields $\chi^i, \chi^{\bar{\imath}}$ respectively, $g_{i\bar{\jmath}}$ is the Kaehler metric of the scalar-fields σ-model and $S(z^i, z^{\bar{\imath}}) \equiv W(z)e^{\frac{K}{2}}$ is the gravitino mass,

W being the superpotential and $K(z^i, z^{\bar{\imath}})$ the Kaehler potential. [9] From the previous parametrizations one derives the supersymmetry transformation laws:

$$\begin{aligned} \delta_\epsilon V^a &= i\bar{\epsilon}\gamma^a\Psi\,, &(5.5)\\ \delta_\epsilon \Psi &= \nabla\epsilon + L_a\,\gamma^{ab}\epsilon V_b + \left[(\mathrm{Re}\,S) + \mathrm{i}\gamma^5(\mathrm{Im}\,S)\right]\gamma^a\epsilon V_a\,. &(5.6)\end{aligned}$$

To define the entropy current we use the notation $a, b, c, \ldots$ for Lorentz vector indices and $I, J, K, \ldots$ for 3-dimansional spatial indices while Greek indices $\alpha, \beta, \gamma, \ldots$ will denote the spinor-component indices of the gravitino 1-form, running from 1 to 4.

Inspired by eq. (4.16), we define the following entropy current in the presence of supergravity:

$$\boxed{J_{(3|4)} = \frac{1}{3!}\,\epsilon_{I_1\ldots I_3}V^{I_1}\wedge V^{I_2}\wedge V^{I_3}\bigwedge_{\beta=1}^{4}\delta(\Psi^\beta)} \tag{5.7}$$

where V^I are the vielbeins (with spatial indices) and Ψ^β are the spinor components of the gravitino 1-form. Note that $J_{(3|4)}$ is a 3-form with picture-number 4. We now show that it is (covariantly) closed. Indeed

$$\begin{aligned}\nabla J_{(3|4)} &= \frac{\mathrm{i}}{4}\,\epsilon_{I_1I_2I_3}\,\bar{\Psi}\wedge\gamma^{I_1}\Psi\wedge V^{I_2}\wedge V^{I_3}\bigwedge_{\beta=1}^{4}\delta(\Psi^\beta) + \\ &+ \frac{2}{3}\,\epsilon_{I_1I_2I_3}V^{I_1}\wedge V^{I_2}\wedge V^{I_3}\wedge\left[\sum_{\beta=1}^{4}\delta'(\Psi^\beta)\wedge\nabla\Psi^\beta\bigwedge_{\alpha\neq\beta}\delta(\Psi^\alpha)\right]\,, \qquad (5.8)\end{aligned}$$

where we used the constraint (5.3), implying $dV^I - \omega^I_a\wedge V^a = \frac{\mathrm{i}}{2}\,\bar{\Psi}\gamma^I\Psi$.

Because of the presence of the current $\bar{\Psi}\gamma^I\Psi$, the first line of eq. (5.8) actually vanishes in force of the identity $\Psi^\alpha\delta(\Psi^\alpha) = 0$. As far as the second term is concerned, we observe that substituting to $\nabla\Psi^\beta = \rho^\beta$ the right hand side of equation (5.4), we get four kinds of contributions: the terms with five vierbeine identically vanish in four dimensions; as far as the contribution of the other three terms is concerned, they contain the products $\gamma_5\gamma^{ab}\,\Psi\wedge V_b$, $\gamma^a\Psi\wedge V_a$ and $\gamma_5\gamma^a\Psi\wedge V_a$. All of them, however, give a vanishing contribution owing to the traceless property of the γ-matrix algebra. Take for example the term

$$-\mathrm{i}(\mathrm{Im}\,S)\gamma^a\gamma^5\Psi\wedge V_a.$$

We have:

$$\begin{aligned} &-\mathrm{i}(\mathrm{Im}\,S)\epsilon_{I_1\ldots I_3}V^{I_1}\wedge V^{I_2}\wedge V^{I_3}\wedge V_a\left[\sum_{\beta=1}^{4}\delta'(\Psi^\beta)\wedge(\gamma^a\,\gamma^5)^\beta{}_\gamma\,\Psi^\gamma\bigwedge_{\alpha\neq\beta}\delta(\Psi^\alpha)\right] \\ &= \mathrm{i}(\mathrm{Im}\,S)\epsilon_{I_1\ldots I_3}V^{I_1}\wedge V^{I_2}\wedge V^{I_3}\wedge V_a\left[\sum_{\beta=1}^{4}(\gamma^a\,\gamma^5)^\beta{}_\beta\bigwedge_{\alpha}^{4}\delta(\Psi^\alpha)\right] = 0\,, \qquad (5.9)\end{aligned}$$

where we used the property $\delta'(\Psi^\gamma)\Psi^\gamma = -\delta(\Psi^\gamma)$.

By the same argument one finds that also the terms proportional to $\gamma^a\Psi$ and $\gamma^5\gamma^{ab}\Psi$ give vanishing contribution since $Tr(\gamma^5\gamma^{ab}) = Tr(\gamma^a) = 0$.

[9] Note that in this formalism $\rho_{ab}V^a_\mu V^b_\nu$ is the supercovariant gravitino field-strength while $\rho_{\mu\nu}$ is the ordinary field-strength $\nabla_{[\mu}\Psi_{\nu]}$.

In conclusion:

$$\nabla J_{(3|4)} = 0\,. \tag{5.10}$$

In order to compare formula (5.7) with the flat case, we decompose the vielbein V^a and the gravitino Ψ^α into flat directions as follows

$$V^a = V^a_\mu\, dx^\mu + V^a_\alpha\, d\theta^\alpha\,, \quad \Psi^\alpha = \Psi^\alpha_\mu\, dx^\mu + \Psi^\alpha_\beta\, d\theta^\beta\,. \tag{5.11}$$

The components can be cast into a supermatrix $E^A_M = \begin{pmatrix} V^a_\mu & V^a_\alpha \\ \Psi^\alpha_\mu & \Psi^\alpha_\beta \end{pmatrix}$ since the components $V^a_\mu, \Psi^\alpha_\beta$ are bosonic, while $V^a_\alpha, \Psi^\alpha_\mu$ are fermionic. Then, we have

$$\begin{aligned} J_{(3|4)} &= \frac{1}{3!}\epsilon_{I_1\dots I_3}(V^{I_1}_{\mu_1}\, dx^{\mu_1} + V^{I_1}_{\alpha_1}\, d\theta^{\alpha_1})\ \wedge (V^{I_2}_{\mu_1}\, dx^{\mu_2} + V^{I_2}_{\alpha_2}\, d\theta^{\alpha_2}) \wedge \\ &\quad \wedge (V^{I_3}_{\mu_3}\, dx^{\mu_3} + V^{I_3}_{\alpha_3}\, d\theta^{\alpha_3})\ \wedge \delta(\Psi^{\alpha_1}_{\mu_1}\, dx^{\mu_1} + \Psi^{\alpha_1}_{\beta_1}\, d\theta^{\beta_1}) \wedge \\ &\quad \cdots \wedge \delta(\Psi^{\alpha_4}_{\mu_4}\, dx^{\mu_4} + \Psi^{\alpha_4}_{\beta_4}\, d\theta^{\beta_4}) \end{aligned} \tag{5.12}$$

The Dirac delta functions should be expanded to the highest order as follows $\delta(d\theta^\alpha + F^\alpha_\mu dx^\mu) = \sum_{k=0}^4 \delta^{(k)}(d\theta^\alpha) \bigwedge_{l=0}^k F^{\alpha_l}_{\mu_l} dx^{\mu_l}$ where $\delta^{(k)}(x)$ are the derivative of the Dirac delta function. Using the rule $x\delta'(x) = -\delta(x)$ (see Appendix A) the above formula reduces to

$$J_{(3|4)} = E\, dx^1 \wedge \cdots \wedge dx^3 \wedge \delta(d\theta^1) \wedge \cdots \wedge \delta(d\theta^4) \tag{5.13}$$

where E is the superdeterminant of the matrix E^A_M.

5.2 Extended Supergravities

In this Appendix we show that the covariant closure of the entropy current, shown to be satisfied for $\mathcal{N} = 1$, $D = 4$ supergravity coupled to chiral multiplets, actually holds for N-extended matter coupled supergravities as well.

For this purpose it is more useful and efficient to use the formulation in terms of Weyl -projected spinors, which, in particular, give a manifest formulation of the R-symmetry U(N) of the theory.

Let us first reformulate the procedure of Section 7 in this new setting, since this allows a straightforward extension to higher N . Using Weyl spinors, the left- handed and right-handed projection of the Majorana gravitino 1-form Ψ will be denoted as $\Psi_\bullet$, $\Psi^\bullet$ respectively. Then equations from (5.1) to (5.5) now take the following form:

$$\begin{aligned} T^a &= dV^a - \omega^a{}_b \wedge V^b - i\,\bar\Psi_\bullet \gamma^a \Psi^\bullet\,, \\ \rho_\bullet &= d\Psi_\bullet - \frac{1}{4}\gamma^{ab}\omega_{ab}\Psi_\bullet \equiv \nabla\Psi_\bullet\,, \\ \rho^\bullet &= d\Psi^\bullet - \frac{1}{4}\gamma^{ab}\omega_{ab}\Psi^\bullet \equiv \nabla\Psi^\bullet, \end{aligned} \tag{5.14}$$

$$T^a = 0\,, \tag{5.15}$$

$$\rho_\bullet = \rho_{\bullet\, ab} V^a \wedge V^b + \mathrm{L}_a\, \gamma^{ab}\Psi_\bullet \wedge V^b + S(z,\bar z)\gamma^a\, \Psi^\bullet \wedge V_a\,, \tag{5.16}$$

$$\rho^\bullet = \rho^{\bullet\, ab} V^a \wedge V^b + \mathrm{L}^*{}_a\, \gamma^{ab}\Psi^\bullet \wedge V^b + S^*(z,\bar z)\gamma^a\, \Psi_\bullet \wedge V_a. \tag{5.17}$$

$$\begin{aligned}\delta_\epsilon V^a &= i\left(\bar{\epsilon}_\bullet \gamma^a \Psi^\bullet - \bar{\epsilon}^\bullet \gamma^a \Psi_\bullet\right) ,\\ \delta_\epsilon \Psi_\bullet &= \nabla \epsilon_\bullet + L_a\, \gamma^{ab} \Psi_\bullet \wedge V_b + S\gamma^a\, \epsilon^\bullet \wedge V_a \end{aligned} \tag{5.18}$$

The entropy current in the presence of supergravity is now defined as follows

$$\boxed{J_{(3|4)} = \frac{1}{3!}\,\epsilon_{I_1\ldots I_3} V^{I_1} \wedge V^{I_2} \wedge V^{I_3} \bigwedge_{\beta=1}^{2} \delta(\Psi_{\bullet\beta}) \wedge \delta(\Psi^{\bullet\beta})}\,, \tag{5.19}$$

where V^I are the vielbeins (with spatial indices) while $\Psi_{\bullet\alpha}, \Psi^{\bullet\beta}$, are the left- and right-handed gravitino components. We notice that now upper and lower Greek indices $\alpha, \beta, \ldots$ of the left- and right-handed spinor components, respectively, run over only two values. The proof of the covariant closure of $J_{(3|4)}$ now goes as follows.

$$\begin{aligned}\nabla J_{(3|4)} &= \frac{1}{2}\,\epsilon_{I_1\ldots I_3} i\, \bar{\Psi}_\bullet \wedge \gamma^{I_1}\Psi^\bullet \wedge V^{I_2} \wedge V^{I_3} \bigwedge_{\beta=1}^{2} \delta(\Psi_{\bullet\beta}) \wedge \delta(\Psi^{\bullet\beta}) + \\ &+ \frac{1}{3}\,\epsilon_{I_1 I_2 I_3} V^{I_1} \wedge V^{I_2} \wedge V^{I_3} \wedge \left[\sum_{\alpha=1}^{2} \delta'(\Psi_{\bullet\alpha}) \wedge (\nabla\Psi)_{\bullet\alpha} \wedge \delta(\Psi_{\bullet\sigma\neq\alpha}) \bigwedge_{\beta=1}^{2} \delta(\Psi^{\bullet\beta}) + \right. \\ &\left. + \sum_{\beta=1}^{2} \delta'(\Psi^{\bullet\beta}) \wedge \nabla(\Psi^{\bullet\beta}) \wedge \delta(\Psi^{\bullet\sigma\neq\beta}) \bigwedge_{\alpha=1}^{2} \delta(\Psi_{\bullet\alpha})\right]\end{aligned} \tag{5.20}$$

where we have used the constraint $dV^I - \omega^I_a \wedge V^a = i\,\bar{\Psi}^\bullet \gamma^I \Psi_\bullet$. As in the Majorana case the presence of the current $\bar{\Psi}^\bullet\gamma^I\Psi_\bullet$ annihilates the first line of eq. (5.20). As far as the terms in the square brackets are concerned, we again see that, substituting to $\nabla(\Psi_{\bullet\beta})$ and $\nabla(\Psi^{\bullet\beta})$ the right hand side of equations (5.3) and (5.4), there is a term vanishing due to the presence of five vielbeine .
To show that also the other terms on the right hand side of those equations give vanishing contributions, let us focus on the first term in square brackets. There are two kinds of terms in the parametrization of $\rho_\bullet = \nabla\Psi_\bullet$ of the type $\Psi \wedge V^a$: One proportional to $L_a\,\gamma^{ab}\,\Psi_\bullet \wedge V_b$ and the other to $S\gamma^a\Psi^\bullet \wedge V_a$. The latter kind of terms gives vanishing contribution since the components of $\Psi^\bullet$ are annihilated by the complete set of corresponding delta functions multiplying it. As for the former type of terms, it is useful to write the spinorial indices explicitly:

$$\sum_{\alpha=1}^{2} \delta'(\Psi_{\bullet\alpha}) \wedge (\gamma^{ab})_\alpha{}^\sigma\, \Psi_{\bullet\sigma} \wedge \delta(\Psi_{\bullet\beta\neq\alpha}) \cdots = -\sum_{\alpha=1}^{2} (\gamma^{ab})_\alpha{}^\alpha \bigwedge_{\beta=1}^{2} \delta(\Psi_{\bullet\beta}) \cdots\,, \tag{5.21}$$

where the ellipses denote the product over all the right-handed delta functions $\delta(\Psi^\bullet)$. By the same argument the second term in square brackets gives a contribution proportional to $(\gamma^{ab})^\alpha{}_\alpha$.[10] Therefore the whole contribution to the of the $\Psi_\bullet \wedge V$ terms in the parametrization of $\nabla\Psi_\bullet$ to the right hand side of (5.20) turns out to be proportional to:

$$\sum_{\alpha=1}^{2} \left(L_a(\gamma^{ab})_\alpha{}^\alpha + L^*_a(\gamma^{ab})^\alpha{}_\alpha\right) \prod_{\beta=1}^{2} \delta(\Psi_{\bullet\beta})\delta(\Psi^{\bullet\beta})\,, \tag{5.22}$$

[10] Notice that in the spinorial notation, the two 2×2 matrices $(\gamma^{ab})_\alpha{}^\beta$ and $(\gamma^{ab})^\alpha{}_\beta$ denote the action, modulo a coefficient, of the Lorentz generators on the left-handed and right-handed spinors respectively. They define the diagonal blocks of the Lorentz generators on the four-component Dirac spinors.

which is identically zero being proportional to the trace of gamma-matrices (actually Pauli matrices).

Given these preliminaries the extension of the above argument to $\mathcal{N}$- extended supergravities is immediate. In this case the $\bullet$ subscripts and superscripts on Ψ are replaced by R–symmetry indices $A, B = 1, \ldots, \mathcal{N}$: $\Psi^{\bullet} \to \Psi^{A}$, $\Psi_{\bullet} \to \Psi_{A}$. The definition of the entropy current is straightforwardly generalized to:

$$\boxed{J_{(3|4\mathcal{N})} = \frac{1}{3!}\,\epsilon_{I_1\ldots I_3} V^{I_1} \wedge V^{I_2} \wedge V^{I_3} \wedge \bigwedge_{A=1}^{\mathcal{N}} \bigwedge_{\alpha=1}^{2} \delta(\Psi_{A\alpha}) \wedge \delta(\Psi^{A\alpha})\,,} \tag{5.23}$$

and eq.s (5.16) and (5.17) are given by

$$\rho_A = \rho_{A\,ab} V^a \wedge V^b + \mathrm{L}_{aA}{}^{B}\,\left(\gamma^{ab} + \eta^{ab}\right)\Psi_B \wedge V_b + T_{[AB]\,ab}\,\gamma^b \Psi^B \wedge V_a + S_{AB}\gamma^a\,\Psi^B \wedge V_a\,, \tag{5.24}$$

$$\rho^A = \rho^{A\,ab} V^a \wedge V^b + \mathrm{L}_{a}{}^{A}{}_{B}\,\left(\gamma^{ab} + \eta^{ab}\right)\Psi^B \wedge V_b + T^{[AB]}_{ab}\,\gamma^b \Psi_B \wedge V_a + S^{AB}\gamma^a\,\Psi_B \wedge V_a\,, \tag{5.25}$$

where $\mathrm{L}_{a}{}^{A}{}_{B} = (\mathrm{L}_{aA}{}^{B})^*$ and $S^{(AB)} = (S_{(AB)})^*$ are complex tensors depending on the sigma-model scalar fields, while $T_{[AB]\,ab} = (T^{[AB]}_{ab})^*$ are the graviphoton scalar-dependent field strengths. Since the proof of closure of the entropy current is essentially based on the properties of the Clifford algebra, we see that, compared to the $\mathcal{N} = 1$ case, the main difference is that the γ^{ab} matrix has been replaced by $\gamma^{ab} + \eta^{ab}$. Then proceeding along the same lines as in the $\mathcal{N} = 1$ case, computing the covariant derivative of $J^{(3|4\mathcal{N})}$, one finds that the only non trivially vanishing term is:

$$\sum_{a\,,\alpha} \left[\mathrm{L}_{a|A}{}^{A}\,\left(\gamma^{ab} + \eta^{ab}\right))_{\alpha}{}^{\alpha} + \mathrm{L}_{a|}{}^{A}{}_{A}\,\left(\gamma^{ab} + \eta^{ab}\right){}^{\alpha}{}_{\alpha}\right] \tag{5.26}$$

Besides the vanishing of the γ^{ab} traces the terms proportional to η^{ab} also vanish since, owing to the fact that $\mathrm{L}_{a|A}{}^{A}$ is a purely imaginary quantity

$$\mathrm{L}_{a|A}{}^{A} + \mathrm{L}_{a|}{}^{A}{}_{A} = 0. \tag{5.27}$$

Finally we observe that exactly the same mechanism also works for higher dimensional supergravities: Indeed the structure of the constraints in superspace for the gravitino field strength is strictly analogous to equations (5.24) and (5.25)the difference being now in the set of γ matrices in D- dimensions appearing in the gravitino field-strengths. The general higher dimensional formula replacing (5.24) and (5.25) has the following analogous form

$$\rho_A = \rho_{A\,ab} V^a \wedge V^b + \sum_k T^{(k)}_{AB|a_1\ldots a_{i_k}} \Delta^{aa_1\ldots a_{i_k}} \Psi^B \wedge V_a + \sum_k T^{(k)}_{abc\ldots} \Delta^{abc\ldots d} \Psi_A \wedge V_d, \tag{5.28}$$

and an analogous expression for ρ^A. Here the matrices $\Delta^{ab\cdots}$ are appropriate linear combinations of the $D+1$ dimensional Clifford algebra of the Γ-matrices , $T^{(k)}_{AB|a_1\ldots a_{i_k}}$ and $T^{(k)}_{abc\ldots}$ are the components of k-forms in $D+1$ dimensions and $\{a, b, \ldots\} = \{0, 1, \cdots D\}$. As we have seen in the previous example the vanishing of the covariant differential of the entropy current only depends of the trace of any element of a complete set of Γ- matrices, implying that the same happens in our case for the Δ-matrices.

6 Conclusions

We develop a new formalism for the entropy current in the framework of supersymmetric theories. In the second part of the paper, we consider the application of the formalism to supergravity and this suggests a possible interpretation and an interesting future direction to be explored: a solution to the supergravity equations is certainly the flat superspace yielding the entropy current for the flat-space variables obtained in the first chapters. Therefore, the comoving coordinates of the fluid provide the manifold where formulating the supergravity; in that context, we are tempted to see the supergravity emerging from the fluid dynamic. The entropy formula for supergravity might give us the opportunity to extend the well-known theorem about the non-decreasing of the divergence of the entropy in gravity.

As mentioned earlier, our approach can be generalized to provide a (supersymmetric) field theoretical description of the superfluid phase, using an explicit dependence of the Lagrangian on the variables X, Z, B^{IJ} introduced in Sect. 2. A note on this point will appear shortly [19].

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A Appendix A

A.1 Conventions

We generally use $\mu, \nu, \cdots = 0, 1, \cdots, d$ to denote space-time indices and $\alpha, \beta, \cdots = 1, \cdots, m$ to denote spinor indices. We adopt the "mostly plus" signature of the metric and the following definition of the Hodge dual operation in $D = d + 1$ space-time dimensions:

$$^\star(dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k}) = \frac{1}{(D-k)!\sqrt{|g|}}\, \epsilon^{\mu_1 \cdots \mu_k \mu_{k+1} \cdots \mu_D}\, dx_{\mu_{k+1}} \wedge \cdots \wedge dx_{\mu_D}\,. \tag{A.1}$$

In this convention we have:

$$^{\star\star}\omega^{(p)} = (-1)^{p(D-p)+1}\, \omega^{(p)} \quad , \quad \omega^{(p)} \wedge {}^\star\eta^{(p)} = -\frac{\sqrt{|g|}}{p!}\, \omega_{\mu_1 \ldots \mu_p}\, \eta^{\mu_1 \ldots \mu_p}\, d^D x\,, \tag{A.2}$$

where we have used, for a generic p–form $\omega^{(p)}$, the following representation:

$$\omega^{(p)} = \frac{1}{p!}\, \omega_{\mu_1 \ldots \mu_p}\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}\,. \tag{A.3}$$

We also use the convention:

$$dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_D} = \epsilon^{\mu_1 \ldots \mu_D}\, d^D x\,. \tag{A.4}$$

A.2 Properties of the Integral Forms

In this section we briefly recall the definition of "integral forms" and their main properties referring mainly to [15] for a detailed exposition.

The problem is that we can build the space Ω^k of k-superforms out of basic 1-superforms $d\theta^\alpha$ and their wedge products, however these products are necessarily commutative, since the θ_α's are odd variables. Therefore, together with the differential operator d, the spaces Ω^k form a differential complex

$$0 \xrightarrow{d} \Omega^0 \xrightarrow{d} \Omega^1 \ldots \xrightarrow{d} \Omega^n \xrightarrow{d} \ldots \tag{A.5}$$

which is bounded from below, but not from above.

One simple way to define the "integral forms" is to introduce a new sheaf containing, among other object to be defined, new basic forms $\delta(d\theta)$. We think of $\delta(d\theta)$ as an operator acting formally on the space of superforms as the usual Dirac's delta measure. We write this as

$$\langle f(d\theta), \delta(d\theta)\rangle = f(0),$$

where f is a superform. Moreover we consider more general objects such as the derivatives $\delta^{(n)}(d\theta)$. Here we have

$$\left\langle f(d\theta), \delta^{(n)}(d\theta)\right\rangle = -\left\langle f'(d\theta), \delta^{(n-1)}(d\theta)\right\rangle = (-1)^n f^{(n)}(0),$$

like the usual Dirac δ measure. Moreover we can consider objects such as $g(d\theta)\delta(d\theta)$, which act by first multiplying by g then applying $\delta(d\theta)$ (in analogy with a measure of type $g(x)\delta(x)$), and so on. The formal properties above imply in addition some simple relations:

$$\delta(d\theta) \wedge \delta(d\theta') = -\delta(d\theta') \wedge \delta(d\theta), \; d\theta \wedge \delta(d\theta) = 0, \; d\theta \wedge \delta'(d\theta) = -\delta(d\theta). \tag{A.6}$$

The systematic exposition of these rules can be found in [14]. An interesting consequence of this procedure is the existence of "negative degree" forms, which are those which reduce the degree of forms (e.g. $\delta'(d\theta)$ has degree -1).

We introduce also the *picture number* by counting the number of delta functions (and their derivatives) and we denote by $\Omega^{r|s}$ the r-forms with picture s. For example the integral form

$$dx^{[\mu_1} \wedge \ldots \wedge dx^{\mu_l]} \wedge d\theta^{(\alpha_1} \wedge \ldots \wedge d\theta^{\alpha_r)} \wedge \delta(d\theta^{[\alpha_{r+1}}) \wedge \ldots \wedge \delta(d\theta^{\alpha_{r+s}]}) \tag{A.7}$$

is an r-from with picture s. All indices μ_i and $\alpha_{r+1}, \ldots \alpha_{r+s}$ are antisymmetrized while $\alpha_1, \ldots \alpha_r$ are symmetrized. Indeed, by also adding derivatives of delta forms $\delta^{(n)}(d\theta)$, even negative form-degree can be considered, e.g. a form of the type:

$$\delta^{(n_1)}(d\theta^{\alpha_1}) \wedge \ldots \wedge \delta^{(n_s)}(d\theta^{\alpha_s}) \tag{A.8}$$

is a $-(n_1 + \ldots n_s)$-form with picture s. Clearly $\Omega^{k|0}$ is just the set Ω^k of superforms, for $k \geq 0$.

We can formally expand the Dirac delta functions in series

$$\delta\left(d\theta^1 + d\theta^2\right) = \sum_j \frac{\left(d\theta^2\right)^j}{j!} \delta^{(j)}(d\theta^1) \tag{A.9}$$

Recall that any usual superform is a polynomial in the $d\theta$, therefore only a finite number of terms really matter in the above sum, when we apply it to a superform. Infact, applying the formulae above, we have for example,

$$\left\langle (d\theta^1)^k, \sum_j \frac{(d\theta^2)^j}{j!}\delta^{(j)}(d\theta^1) \right\rangle = (-1)^k (d\theta^2)^k \tag{A.10}$$

Notice that this is equivalent to the effect of replacing $d\theta^1$ with $-d\theta^2$. We could have also interchanged the role of θ^1 and θ^2 and the result would be to replace $d\theta^2$ with $-d\theta^1$. Both procedures correspond precisely to the action we expect when we apply the $\delta\left(d\theta^1 + d\theta^2\right)$ Dirac measure.

The integral forms form a new complex as follows

$$\ldots \xrightarrow{d} \Omega^{(r|q)} \xrightarrow{d} \Omega^{(r+1|q)} \ldots \xrightarrow{d} \Omega^{(p+1|q)} \xrightarrow{d} 0 \tag{A.11}$$

where $\Omega^{(p+1|q)}$ is the top "form" $dx^{[\mu_1} \wedge \ldots \wedge dx^{\mu_{p+1}]} \wedge \delta(d\theta^{[\alpha_1}) \wedge \ldots \wedge \delta(d\theta^{\alpha_q]})$ which can be integrated on the supermanifold. As in the usual commuting geometry, there is an isomorphism between the cohomologies $H^{(0|0)}$ and $H^{(p+1|q)}$ on a supermanifold of dimension $(p+1|q)$. In addition, one can define two operations acting on the cohomology groups $H^{(r|s)}$ which change the picture number s (see [14]).

Given a function $f(x,\theta)$ on the superspace, we define its integral by the super top-form $\omega^{(p+1|q)} = f(x,\theta) d^{p+1}x \delta(d\theta^1) \ldots \delta(d\theta^q)$ belonging to $\Omega^{(p+1|q)}$ as follows

$$\int_{\mathbb{R}^{(p+1|q)}} \omega^{(p+1|q)} = \frac{1}{q!}\epsilon^{\alpha_1 \ldots \alpha_q} \partial_{\theta^{\alpha_1}} \ldots \partial_{\theta^{\alpha_q}} \int_{\mathbb{R}^{p+1}} f(x,\theta) d^{p+1}x \tag{A.12}$$

where the last equalities is obtained by integrating on the delta functions and selecting the bosonic top form. The remaining integrals are the usual integral of densities and the Berezin integral. It is easy to show that indeed the measure is invariant under general coordinate changes and the density transform as a Berezinian with the superdeterminant. Note that in particular we have

$$\int \bigwedge_{\alpha=1}^{m} \theta^\alpha \, \delta(d\theta^\alpha) = 1\,. \tag{A.13}$$

A.3 $\star$-Operator for Supermanifolds

Here we define a $\star$-operator for the supermanifolds. First we briefly review the construction of a $\star$-operator for a manifold $\mathcal{M}$, we consider two p-forms ω^p, η^p and we define

$$(\omega, \eta) = \int_{\mathcal{M}} \omega \wedge {}^\star\eta = \int_{\mathcal{M}} \langle \omega, \eta \rangle Vol_{n+1} \tag{A.14}$$

where $\langle \omega, \eta \rangle$ is define as local product of two p-forms

$$\langle \omega, \eta \rangle = \frac{1}{n!} g^{\mu_1\nu_1} \ldots g^{\mu_n\nu_n} \omega_{\mu_1 \ldots \mu_n} \eta_{\nu_1 \ldots \nu_n}\,, \tag{A.15}$$

usually also denoted as Gram determinant. $g^{\mu\nu}$ are the components of the matrix which is the inverse of metric tensor g on $\mathcal{M}$.

Our aim is to define the Hodge star operator in a Grassmann manifold. We try to do that in such a way as to reproduce formula (A.12) for the integration of the top form.

Given a function $f(\theta,\, d\theta, \delta(d\theta))$ of the fermionic variables, their differential and the delta-functions thereof, we define its dual in the supermanifold as the function ${}^*f(\theta,\, d\theta, \delta(d\theta))$ satisfying the property:

$$\int f(\theta,\, d\theta, \delta(d\theta)) \wedge {}^*f(\theta,\, d\theta, \delta(d\theta)) = \int dV^{(Gr)} = 1\,. \tag{A.16}$$

where we have used the following definition of the volume element in the Grassmann space of dimensions m:

$$dV^{(Gr)} = \bigwedge_{\alpha=1}^{m} \theta^\alpha\, \delta(d\theta^\alpha)\,, \tag{A.17}$$

satisfying $ddV^{(Gr)} = 0$. We also extend the definition of Hodge star operator to the product of a bosonic form $\omega^{(p)}(x)$ times a function $f(\theta,\, d\theta, \delta(d\theta))$ to be:

$${}^*\left[\omega^{(p)}(x) \wedge f(\theta,\, d\theta, \delta(d\theta))\right] = {}^*\omega^{(p)}(x) \wedge {}^*f(\theta,\, d\theta, \delta(d\theta))\,. \tag{A.18}$$

Condition (A.16) allows us to define the Hodge star operator for a $d\theta$-differential in a Grassmann space of dimension m:

$${}^*\, d\theta^\alpha = (-1)^{m+1}\theta^1 \cdots \theta^m \,\frac{\partial}{\partial(d\theta^\alpha)} \bigwedge_{\beta=1}^{m} \delta(d\theta^\beta). \tag{A.19}$$

It follows

$$d\theta^\beta \wedge {}^*\, d\theta^\alpha = -\theta^1 \cdots \theta^m\; d\theta^\beta \frac{\partial}{\partial(d\theta^\alpha)} \bigwedge_{\gamma=1}^{m} \delta(d\theta^\gamma)\,. \tag{A.20}$$

We note that both the indices β and α must coincide with some index in the product; then extracting the $\delta(d\theta^\alpha)$ from the product and considering the factor

$$d\theta^\beta \frac{\partial \delta(d\theta^\alpha)}{\partial(d\theta^\alpha)} \bigwedge_{\gamma\neq\alpha} \delta(d\theta^\gamma)\,,$$

we must also have $\alpha = \beta$ since any other choice is annihilated against $\bigwedge_{\gamma\neq\alpha} \delta(d\theta^\gamma)$. Hence, using the rule

$$d\theta^\beta \frac{\partial \delta(d\theta^\beta)}{\partial(d\theta^\beta)} = -\delta(d\theta^\beta)\,,$$

we find

$$d\theta^\beta \wedge {}^*\, d\theta^\alpha = \delta^{\beta\alpha}\, \theta^1 \cdots \theta^m \bigwedge_{\gamma=1}^{m} \delta(d\theta^\gamma) = \delta^{\beta\alpha}\, dV^{(Gr)}\,. \tag{A.21}$$

Similarly, using the definition (A.16), it is straightforward to show that

$${}^\star\theta^\alpha = \frac{\partial}{\partial\theta^\alpha}\, \left(\theta^1 \cdots \theta^m\right) \bigwedge_{\gamma=1}^{m} \delta(d\theta^\gamma)\,. \tag{A.22}$$

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